

Reflection Equation Algebra of a (h, w) -deformed Oscillator

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Abstract

We consider the reflection equation algebra for a finite dimensional R -matrix for the (h, w) -deformed Heisenberg algebra $\mathcal{U}_{h,w}(h(4))$. A representation of the reflection matrix K is constructed using the matrix generators $L^{(\pm)}$ of the $\mathcal{U}_{h,w}(h(4))$ algebra. A series of representations of the K -matrix then may be generated by using the coproduct rules of the $\mathcal{U}_{h,w}(h(4))$ algebra. The complementary condition necessary for combining two distinct solutions of the reflection equation algebra yields the braiding relations between these two sets of generators. This may be thought as a generalization of Bose-Fermi statistics to braiding statistics, which then may be used to provide a new braided colagebraic structure to a Hopf algebra generated by the elements of the matrix K . The reflection equation algebra and the braided exchange properties are found to depend on both deformation parameters h and w .

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1 Introduction

The representation theory of quantum algebras has led to various deformed oscillator algebras [1-7]. These studies may lead to field theories, where excitations obeying braiding statistics [4,8] may be discussed. Other applications of deformed Heisenberg algebra include the description [9] of a class of exactly solvable potentials in terms of dynamical symmetries, the theory of link invariants [10] and the theory of q -special functions [11]. At an appropriate contraction limit of the quantum algebra $\mathcal{U}_q(sl(2))$, a ‘standard’ deformed Heisenberg algebra $\mathcal{U}_h(h(4))$ and its universal \mathcal{R} -matrix has been obtained [12]. A different ‘nonstandard’ quantization of the Heisenberg algebra $\mathcal{U}_w(h(4))$ has also been obtained [13] as a contraction limit of the Jordanian deformation of $sl(2)$ algebra [14]. A distinction between these two deformations appears in their properties under a rescaling of the Heisenberg generators: $A_{\pm} \rightarrow \lambda^{\pm 1} A_{\pm}$. While the ‘standard’ $\mathcal{U}_h(h(4))$ algebra remains invariant under this rescaling, the ‘nonstandard’ algebra $\mathcal{U}_w(h(4))$ does so only if we also rescale the deformation parameter $w \rightarrow \lambda^{-1} w$. In this sense, the algebra $\mathcal{U}_w(h(4))$ has close kinship with κ -Poincaré algebra [15] and deformed conformal algebra [16], where the introduction of mass-like deformation parameter leads to the appearance of the fundamental mass on basic geometrical level. Combining the features of the ‘standard’ and ‘non-standard’ deformation procedures, a new two-parametric quantization of the Heisenberg algebra $\mathcal{U}_{h,w}(h(4))$ has recently been obtained [17]. The full Hopf structure of the $\mathcal{U}_{h,w}(h(4))$ algebra, its universal \mathcal{R} -matrix and the corresponding matrix pseudo-group have also been discussed. From the point of view of applicability in concrete physical models, the algebra $\mathcal{U}_{h,w}(h(4))$ with multiparameter (h, w) -deformation may be of interest.

In another development, the reflection equation (RE) algebra connected to the quantum group has attracted wide interest. The RE was introduced in [18] to describe factorized scattering on a half-line, and the related algebra soon found quite different applications in quantum current algebras [19], integrable models with non-periodic boundary conditions [20] and in the description [21] of braid group representations on a handlebody. A generalization of the RE algebra for the scattering of lines moving in a half plane touching the boundary in $2 + 1$ -dimension has been made [22]. Recently, there has been renewed interest in this topic following investigations in condensed matter physics in which boundaries play a significant role. The boundary states, which correspond to the lowest lying energy states in integrable field theories and statistical mechanics models with boundaries have been explored [23,24]. In an attempt to establish a second quantized approach to this problem, an oscillator algebra and the associated Fock space satisfying reflection boundary condition and obeying generalized statistics has been studied [25]. In this connection it is of interest to study the RE algebra corresponding to the R -matrix of the doubly deformed Heisenberg algebra $\mathcal{U}_{h,w}(h(4))$. The braided group approach [26,27] to the RE algebras takes the point of view that in the exchanges between two copies of the RE algebras, the usual transposition map is replaced by a more general braiding. From this point of view, the braided group corresponding to the singly deformed Heisenberg algebra $\mathcal{U}_h(h(4))$ with the ‘standard’

deformation parameter h has been studied [28]. In the present context, it may be useful to investigate the braided exchange properties described by the RE algebra of the (h, w) -deformed oscillator. In particular, the role of the ‘nonstandard’ deformation parameter w in the structure of the braiding statistics is worth exploring. This may serve as a pointer towards understanding the role of the dimensional mass-like deformation parameters in the context of RE algebras corresponding to the κ -deformed Poincaré algebra [15] and deformed conformal algebras [16].

In this article, we study the spectral parameter independent form of the RE algebra related to the quantum algebra $\mathcal{U}_{h,w}(h(4))$. In Section 2, we review the Hopf structure of the doubly deformed algebra $\mathcal{U}_{h,w}(h(4))$ and recast it using the FRT prescription [29]. The extended RE algebra related to its R -matrix is described in Section 3. The braiding properties of the two copies of the generators of the RE algebra depend on both the deformation parameters (h, w) . We conclude in Section 4.

2 The Hopf Algebra $\mathcal{U}_{h,w}(h(4))$ and its FRT Construction

We start by enlisting the full Hopf structure [17] of the universal enveloping algebra $\mathcal{U}_{h,w}(h(4))$ generated by the elements A_{\pm} , N and E . The algebra reads

$$\begin{aligned} [N, A_+] &= \frac{e^{wA_+} - 1}{w}, & [N, A_-] &= -A_-, \\ [A_-, A_+] &= \frac{\sinh(hE)}{h} e^{wA_+}, & [E, \bullet] &= 0, \end{aligned} \quad (2.1)$$

where E is a central element of the algebra. Another nonlinear central element C exists [17] and may be regarded as the Casimir element of the algebra (2.1):

$$C = \frac{\sinh hE}{h} N + \frac{e^{-wA_+} - 1}{2w} A_- + A_- \frac{e^{-wA_+} - 1}{2w}. \quad (2.2)$$

The coalgebraic structure of $\mathcal{U}_{h,w}(h(4))$ is given by [17]

$$\begin{aligned} \Delta(A_+) &= A_+ \otimes 1 + 1 \otimes A_+, \\ \Delta(A_-) &= A_- \otimes e^{hE} e^{wA_+} + e^{-hE} \otimes A_- + w e^{-hE} N \otimes \frac{\sinh(hE)}{h} e^{wA_+}, \\ \Delta(N) &= N \otimes e^{wA_+} + 1 \otimes N, & \Delta(E) &= E \otimes 1 + 1 \otimes E, \end{aligned} \quad (2.3)$$

$$\varepsilon(A_+) = \varepsilon(A_-) = \varepsilon(N) = \varepsilon(E) = 0, \quad (2.4)$$

$$\begin{aligned} S(A_+) &= -A_+, & S(A_-) &= -A_- e^{-wA_+} + w \frac{\sinh(hE)}{h} N e^{-wA_+}, \\ S(N) &= -N e^{-wA_+}, & S(E) &= -E. \end{aligned} \quad (2.5)$$

Here we note that there exists an invertible map of the algebra (2.1) on the undeformed Heisenberg algebra:

$$a_+ = \frac{1 - e^{-wA_+}}{w}, \quad a_- = \frac{hE}{\sinh(hE)} A_-, \quad n = N, \quad e = E, \quad (2.6)$$

where the generators (a_{\pm}, n, e) are classical in nature:

$$[n, a_{\pm}] = \pm a_{\pm}, \quad [a_-, a_+] = e, \quad [e, \bullet] = 0. \quad (2.7)$$

The universal \mathcal{R} -matrix of $\mathcal{U}_{h,w}(h(4))$ algebra reads [17]

$$\mathcal{R} = e^{-wA_+ \otimes N} e^{-2h E \otimes N} \exp\left(2h e^{hE} A_- \otimes \left(\frac{1 - e^{-wA_+}}{w}\right)\right) e^{wN \otimes A_+}. \quad (2.8)$$

Following the FRT prescription [29], we now recast the Hopf structure of the $\mathcal{U}_{h,w}(h(4))$ algebra in order to explicitly obtain the Lax operators $L^{(\pm)}$, which may be used to construct a hierarchy of solutions [21] of the RE algebra. To this end, we first obtain the conjugate universal $\tilde{\mathcal{R}}$ matrix that satisfies Yang-Baxter equation and is defined as $\tilde{\mathcal{R}} = \mathcal{R}_+^{-1}$, where $\mathcal{R}_+ = P\mathcal{R}P$ and P is the transposition operator:

$$\tilde{\mathcal{R}} = e^{-wA_+ \otimes N} \exp\left(-2h \left(\frac{1 - e^{-wA_+}}{w}\right) \otimes e^{hE} A_- \right) e^{2h N \otimes E} e^{wN \otimes A_+}. \quad (2.9)$$

A real 3×3 -matrix representation of the algebra (2.1) remains undeformed and reads as follows:

$$\begin{aligned} \pi_3(A_+) &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, & \pi_3(A_-) &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ \pi_3(N) &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \pi_3(E) &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned} \quad (2.10)$$

Using the representation (2.10) in one of the sectors of the tensor products in the expressions of \mathcal{R} and $\tilde{\mathcal{R}}$ in (2.8) and (2.9) respectively, the matrix operators $L^{(\pm)}$ may be directly read:

$$\begin{aligned} (\pi_3 \otimes \text{id})\mathcal{R} &= \begin{pmatrix} 1 & \frac{2h}{w}(e^{wA_+} - 1) & -2hN \\ 0 & e^{wA_+} & -wN \\ 0 & 0 & 1 \end{pmatrix} = S(L^{(-)}), \\ (\text{id} \otimes \pi_3)\mathcal{R} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-2hE - wA_+} & 2he^{-hE - wA_+} A_- + we^{-2hE - wA_+} N \\ 0 & 0 & 1 \end{pmatrix} = L^{(+)}, \end{aligned}$$

$$\begin{aligned}
(\pi_3 \otimes \text{id})\tilde{\mathcal{R}} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{2hE+wA_+} & -2he^{hE}A_- - wN \\ 0 & 0 & 1 \end{pmatrix} = S(L^{(+)}) \\
(\text{id} \otimes \pi_3)\tilde{\mathcal{R}} &= \begin{pmatrix} 1 & -\frac{2h}{w}(1 - e^{-wA_+}) & 2he^{-wA_+}N \\ 0 & e^{-wA_+} & we^{-wA_+}N \\ 0 & 0 & 1 \end{pmatrix} = L^{(-)}. \tag{2.11}
\end{aligned}$$

The \mathcal{R} -matrix in the representation (2.10), defined as $R = (\pi_3 \otimes \pi_3)\mathcal{R}$ is given by

$$R = 1 \otimes 1 + 2h (e_{12} \otimes e_{23} - e_{13} \otimes e_{22}) + w (e_{22} \otimes e_{23} - e_{23} \otimes e_{22}) \tag{2.12}$$

where $(e_{ij})_{kl} = \delta_{ik} \delta_{jl}$ and the indices $(i, j, k, l) \in (1, 2, 3)$. The transposed matrix $R_+ = PRP$ and the conjugate matrix $\tilde{R} = R_+^{-1}$ may be obtained readily. The algebra (2.1) may now be reformulated in terms of the matrix generators $L^{(\pm)}$:

$$R_+ L_1^{(\epsilon_1)} L_2^{(\epsilon_2)} = L_2^{(\epsilon_2)} L_1^{(\epsilon_1)} R_+ \tag{2.13}$$

where $L_1^{(\epsilon)} = L^{(\epsilon)} \otimes 1$, $L_2^{(\epsilon)} = 1 \otimes L^{(\epsilon)}$ and $(\epsilon_1, \epsilon_2) = (\pm, \pm), (+, -)$. The coalgebraic properties (2.3)-(2.5) may also be expressed concisely:

$$\triangle(L^{(\epsilon)}) = L^{(\epsilon)} \dot{\otimes} L^{(\epsilon)}, \quad \varepsilon(L^{(\epsilon)}) = 1, \quad S(L^{(\epsilon)}) = (L^{(\epsilon)})^{-1}. \tag{2.14}$$

In the $h \rightarrow 0$ limit, only the ‘nonstandard’ deformation parameter w survives and the Hopf algebra $\mathcal{U}_{h,w}(h(4))$ reduces to $\mathcal{U}_w(h(4))$ discussed in [13]. A feature of the latter is that the matrices \mathcal{R} and $\tilde{\mathcal{R}}$ become identical as is evident from (2.8) and (2.9) in the $h \rightarrow 0$ limit. The above description following the FRT procedure then yields the Lax operator $L_{(h \rightarrow 0)}^{(+)}$ and its antipode. The matrix generator $L_{(h \rightarrow 0)}^{(+)}$ is given by

$$L_{(h \rightarrow 0)}^{(+)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-wA_+} & we^{-wA_+}N \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.15}$$

In the limit $h \rightarrow 0$, the FRT prescription (2.13) and (2.14) only describe the Borel subalgebra generated by the elements A_+ , N and E .

As the dual Hopf algebra $\mathcal{F}un_{h,w}(\mathcal{H}(4))$ is also relevant in discussing the invariance properties of the RE algebra, we briefly discuss it here. Our choice of variables is slightly

different from [17]. The function algebra $\mathcal{F}un_{h,w}(\mathcal{H}(4))$ is generated by the variable elements of the upper triangular matrix

$$T = \begin{pmatrix} 1 & a & b \\ 0 & c & d \\ 0 & 0 & 1 \end{pmatrix}, \quad (2.16)$$

obeying the relation

$$R T_1 T_2 = T_2 T_1 R, \quad (2.17)$$

where $T_1 = T \otimes 1$ and $T_2 = 1 \otimes T$. The algebra reads

$$\begin{aligned} [a, b] &= 2h a + w a^2, & [c, d] &= w c(c-1), & [b, d] &= [a, c] = 0, \\ [a, d] &= w a c, & [b, c] &= -w a c. \end{aligned} \quad (2.18)$$

Assuming that the diagonal element c is invertible, the coalgebra maps are succinctly given by

$$\Delta(T) = T \dot{\otimes} T, \quad \varepsilon(T) = 1, \quad S(T) = T^{-1}. \quad (2.19)$$

In terms of the generators a, b, c and d , the maps (2.19) read

$$\begin{aligned} \Delta(a) &= a \otimes c + 1 \otimes a, & \Delta(b) &= b \otimes 1 + 1 \otimes b + a \otimes d, \\ \Delta(c) &= c \otimes c, & \Delta(d) &= d \otimes 1 + c \otimes d, \\ \varepsilon(a) &= \varepsilon(b) = \varepsilon(d) = 0, & \varepsilon(c) &= 1, \\ S(a) &= -c^{-1} a, & S(b) &= -b + c^{-1} a d, & S(c) &= c^{-1}, & S(d) &= -c^{-1} d. \end{aligned} \quad (2.20)$$

The duality of the Hopf algebras $\mathcal{U}_{h,w}(h(4))$ and $\mathcal{F}un_{h,w}(\mathcal{H}(4))$ may now be expressed by the pairing

$$\langle L^{(\pm)} \otimes 1, 1 \otimes T \rangle = R_{\pm}, \quad (2.21)$$

where $R_- = R^{-1}$.

3 The RE Algebra

Here we study a spectral parameter independent form of RE satisfied by the entries of the reflection matrix K :

$$R K_1 R_+ K_2 = K_2 R K_1 R_+, \quad (3.1)$$

where, as usual, $K_1 = K \otimes 1$ and $K_2 = 1 \otimes K$. The entries of the matrix K are assumed to be in the upper triangular form

$$K = \begin{pmatrix} 1 & \alpha & \beta \\ 0 & \gamma & \delta \\ 0 & 0 & 1 \end{pmatrix}, \quad (3.2)$$

A basic property of the RE algebra (3.1) is its covariance under the quantum group coaction. The transform $K_T = TKT^{-1}$ is also a solution of (3.1) if all the elements of the matrices K and T commute

$$[K_j^i, T_l^k] = 0, \quad (i, j, k, l) \in (1, 2, 3). \quad (3.3)$$

The matrix T satisfies the defining relation (2.17) of the inverse scattering theory. The RE algebra (3.1) requires its generators $(\alpha, \beta, \gamma, \delta)$ to have the following commutation properties:

$$\begin{aligned} [\alpha, \beta] &= \alpha(2h\gamma - w\alpha), & [\alpha, \delta] &= (2h\gamma - w\alpha)(\gamma - 1), \\ [\beta, \delta] &= (2h\gamma - w\alpha)\delta, & [\gamma, \bullet] &= 0. \end{aligned} \quad (3.4)$$

The algebra (3.4) has two central elements

$$C_1 = \gamma, \quad C_2 = \alpha\delta - \beta\gamma + \beta. \quad (3.5)$$

The elements C_1 is the determinant of the reflection matrix K and is assumed to be invertible. This allows us to construct the inverse operator K^{-1} as follows

$$K^{-1} = \begin{pmatrix} 1 & -\gamma^{-1}\alpha & -\beta + \gamma^{-1}\alpha\delta \\ 0 & \gamma^{-1} & -\gamma^{-1}\delta \\ 0 & 0 & 1 \end{pmatrix}. \quad (3.6)$$

Following [30], a representation of the algebra (3.4) may be constructed utilizing the generators of the quantum algebra $\mathcal{U}_{h,w}(h(4))$. Using the commutation rules (2.1), it may be readily checked that the following construction provides a representation of the algebra (3.4):

$$\begin{aligned} K &= S(L^{(-)})L^{(+)} \\ &= \begin{pmatrix} 1 & \frac{2h}{w}e^{-2hE}(1 - e^{-wA_+}) & \frac{4h^2}{w}e^{-hE}(1 - e^{-wA_+})A_- + 2he^{-2hE}(1 - e^{-wA_+})N - 2hN \\ 0 & e^{-2hE} & 2he^{-hE}A_- - 2we^{-hE}\sinh(hE)N \\ 0 & 0 & 1 \end{pmatrix} \end{aligned} \quad (3.7)$$

In representation (3.7), the central elements in (3.5) assume the form

$$C_1 = e^{-2hE}, \quad C_2 = -2he^{-hE}(\sinh(hE) + 2hC), \quad (3.8)$$

where C is the Casimir element (2.2) of the $\mathcal{U}_{h,w}(h(4))$ algebra. The representation (3.7) of the K -matrix becomes trivial in limit $h \rightarrow 0$. This is a consequence of the fact that at $h \rightarrow 0$

limit, the universal \mathcal{R} -matrix in (2.8) and its conjugate matrix $\tilde{\mathcal{R}}$ in (2.9) become identical. The RE algebra (3.4) in this limit, however, remains nontrivial. We will comment on this later.

As observed in [21] the context of $\mathcal{U}_q(sl(2))$ algebra, the construction (3.7) immediately provides a technique to obtain a hierarchy of representations of the RE algebra (3.1). Such representations were used in [21] to construct realizations of the braid groups on handlebodies. In the present context, successive operations of the coproducts on the representation (3.7) of the K -matrix generate a string of solutions of the RE algebra (3.1):

$$\Delta(K_j^i), \quad (\Delta \otimes \text{id}) \circ \Delta(K_j^i), \quad (\Delta \otimes \text{id} \otimes \text{id}) \circ (\Delta \otimes \text{id}) \circ \Delta(K_j^i), \dots, \quad (3.9)$$

where $(i, j) \in (1, 2, 3)$. The coproduct used in (3.9) reflects the coproduct structure (2.14) of the $\mathcal{U}_{h,w}(h(4))$ algebra, where the entries in different sectors are commuting. This contrasts with another coproduct scheme [26-28], where the braiding statistics between different spaces are to be taken into account. Using (2.14) we may successively obtain the representations of the K -matrix, whose entries now assume values in the tensor product spaces of the $\mathcal{U}_{h,w}(h(4))$ algebra. The first few of these representations read as follows:

$$\begin{aligned} \Delta(K_j^i) &= (1 \otimes S(L^{(-)})_l^i)(K_n^l \otimes 1)(1 \otimes L^{(+)}_j^n), \\ (\Delta \otimes \text{id}) \circ \Delta(K_j^i) &= (1 \otimes 1 \otimes S(L^{(-)})_p^i)(1 \otimes S(L^{(-)})_l^p \otimes 1)(K_n^l \otimes 1 \otimes 1) \\ &\quad (1 \otimes L^{(+)}_m^n \otimes 1)(1 \otimes 1 \otimes L^{(+)}_j^m). \end{aligned} \quad (3.10)$$

A key property, important from the point of integrability of models, requires that two independent solutions of the RE algebra satisfying a complementary relation may be combined to construct a new solution of the RE algebra. Namely, if K and K' matrices satisfy (3.1), then the following combinations

$$\hat{K} = KK', \quad \hat{\hat{K}} = KK'K^{-1} \quad (3.11)$$

also obey the same RE algebra provided the following complementary relation holds:

$$RK_1R^{-1}K'_2 = K'_2RK_1R^{-1}. \quad (3.12)$$

The relation (3.12) is covariant under the quantum group coaction. The transforms $K_T = TKT^{-1}$ and $K'_T = TK'T^{-1}$ satisfy (3.12) provided the elements of T commute with all the elements of K and K' . This process of building new solutions may obviously be continued. Equation (3.12) requires the following commutations relations between the elements of K and K' :

$$\begin{aligned} \alpha'\alpha &= \alpha\alpha', & \alpha'\beta &= \beta\alpha' - w\alpha\alpha', & \alpha'\gamma &= \gamma\alpha', & \alpha'\delta &= \delta\alpha' - w(\gamma-1)\alpha', \\ \beta'\alpha &= \alpha\beta' - 2h(\gamma-1)\alpha' + w\alpha\alpha', & \beta'\beta &= \beta\beta' - 2h\delta\alpha' + 2hw(\gamma-1)\alpha', \\ \beta'\gamma &= \gamma\beta', & \beta'\delta &= \delta\beta' - w\delta\alpha' + w^2(\gamma-1)\alpha', \\ \gamma'\alpha &= \alpha\gamma', & \gamma'\beta &= \beta\gamma' & \gamma'\gamma &= \gamma\gamma', & \gamma'\delta &= \delta\gamma', \\ \delta'\alpha &= \alpha\delta' - 2h(\gamma-1)(\gamma'-1) + w\alpha(\gamma'-1), & \delta'\beta &= \beta\delta' - 2h\delta(\gamma'-1) + w\alpha\delta', \\ \delta'\gamma &= \gamma\delta', & \delta'\delta &= \delta\delta' - w\delta(\gamma'-1) + w(\gamma-1)\delta'. \end{aligned} \quad (3.13)$$

The central elements of K and K' are mutually central in both algebras

$$[K_j^i, C'_m] = 0, \quad [K_j^i, C_m] = 0 \quad \text{for} \quad (i, j) \in (1, 2, 3), \quad m \in (1, 2). \quad (3.14)$$

In (3.13) inhomogeneous terms exist for $(\gamma, \gamma') \neq 1$. At this point we wish to draw attention to the fact that the RE algebra (3.4) and the commutation relations (3.13) between two copies of the generators of the algebra (3.4) depend nontrivially on the both the deformation parameters (h, w) . Relations (3.1) and (3.12) taken together are sometimes mentioned as extended RE algebra. The physical meaning of the exchange properties (3.13) become clearer in the language adopted in [26-28], which we discuss subsequently.

In an alternate approach [26,27] to the RE algebra, a function type Hopf algebra $\mathcal{B}(R)$, generated by the elements of the K -matrix, is associated with a R -matrix. The important distinction is that, in the tensor product structure, the usual \pm factor encountered for the twist map of the Hopf superalgebras is replaced by a map ψ ($\neq \psi^{-1}$) characterizing the braid statistics. For $\Gamma_i \in \mathcal{B}(R)$, where for $i = (1, 2, 3, 4)$, the tensor product composition rule reads [26-28]

$$(\Gamma_1 \otimes \Gamma_2)(\Gamma_3 \otimes \Gamma_4) = \Gamma_1 \psi(\Gamma_2 \otimes \Gamma_3) \Gamma_4, \quad (3.15)$$

where the braided transposition generates a linear combination of tensor product terms. Taking this braiding between two copies of the RE algebra into account a new coproduct structure for the K -matrices that is a homomorphism of the RE algebra may be assigned [26-28]. Apart from the extra braiding properties, the new Hopf structure is of the same form as for the matrix quantum group elements in (2.19):

$$\tilde{\Delta}(K) = K \dot{\otimes} K, \quad \tilde{\varepsilon}(K) = 1, \quad \tilde{S}(K) = K^{-1}. \quad (3.16)$$

We repeat here that the braided Hopf structure (3.16) is distinct from the Hopf structure of the $\mathcal{U}_{h,w}(h(4))$ used in (3.9). The latter involves no braiding in the tensor product composition rule.

Expressed in terms of the elements $(\alpha, \beta, \gamma, \delta)$ of the K -matrix, (3.16) assumes the form

$$\begin{aligned} \tilde{\Delta}(\alpha) &= \alpha \otimes \gamma + 1 \otimes \alpha, & \tilde{\varepsilon}(\alpha) &= 0, & \tilde{S}(\alpha) &= -\gamma^{-1} \alpha, \\ \tilde{\Delta}(\beta) &= \beta \otimes 1 + 1 \otimes \beta + \alpha \otimes \delta, & \tilde{\varepsilon}(\beta) &= 0, & \tilde{S}(\beta) &= -\beta + \gamma^{-1} \alpha \delta, \\ \tilde{\Delta}(\gamma) &= \gamma \otimes \gamma, & \tilde{\varepsilon}(\gamma) &= 1, & \tilde{S}(\gamma) &= \gamma^{-1}, \\ \tilde{\Delta}(\delta) &= \delta \otimes 1 + \gamma \otimes \delta, & \tilde{\varepsilon}(\delta) &= 0, & \tilde{S}(\delta) &= -\gamma^{-1} \delta. \end{aligned} \quad (3.17)$$

The braiding rules between two copies of the RE algebra (3.4) may be directly read from (3.13). The primed quantities are entries in the second sector of the tensor product space and quantities like $\Gamma'_1 \Gamma_2$, where $(\Gamma_1, \Gamma_2) \in (\alpha, \beta, \gamma, \delta)$, are interpreted as a way of writing the transpose $\psi(\Gamma_1 \otimes \Gamma_2)$. In these notations (3.13) reads

$$\psi(\alpha \otimes \alpha) = \alpha \otimes \alpha, \quad \psi(\alpha \otimes \beta) = \beta \otimes \alpha - w \alpha \otimes \alpha,$$

$$\begin{aligned}
\psi(\alpha \otimes \gamma) &= \gamma \otimes \alpha, & \psi(\alpha \otimes \delta) &= \delta \otimes \alpha - w (\gamma - 1) \otimes \alpha, \\
\psi(\beta \otimes \alpha) &= \alpha \otimes \beta - 2h (\gamma - 1) \otimes \alpha + w \alpha \otimes \alpha, \\
\psi(\beta \otimes \beta) &= \beta \otimes \beta - 2h \delta \otimes \alpha + 2hw (\gamma - 1) \otimes \alpha, \\
\psi(\beta \otimes \gamma) &= \gamma \otimes \beta, & \psi(\beta \otimes \delta) &= \delta \otimes \beta - w \delta \otimes \alpha + w^2 (\gamma - 1) \otimes \alpha, \\
\psi(\gamma \otimes \alpha) &= \alpha \otimes \gamma, & \psi(\gamma \otimes \beta) &= \beta \otimes \gamma, & \psi(\gamma \otimes \gamma) &= \gamma \otimes \gamma, & \psi(\gamma \otimes \delta) &= \delta \otimes \gamma, \\
\psi(\delta \otimes \alpha) &= \alpha \otimes \delta - 2h (\gamma - 1) \otimes (\gamma - 1) + w \alpha \otimes (\gamma - 1), \\
\psi(\delta \otimes \beta) &= \beta \otimes \delta - 2h \delta \otimes (\gamma - 1) + w \alpha \otimes \delta, \\
\psi(\delta \otimes \gamma) &= \gamma \otimes \delta, & \psi(\delta \otimes \delta) &= \delta \otimes \delta - w \delta \otimes (\gamma - 1) + w (\gamma - 1) \otimes \delta,
\end{aligned} \tag{3.18}$$

The braided exchange properties (3.18) in the $w \rightarrow 0$ limit was earlier discussed in [28] the context of the braided group corresponding to the ‘standard’ deformation of the Heisenberg algebra $\mathcal{U}_h(h(4))$. The central elements (C_1, C_2) in (3.5) are truly bosonic in nature in the sense

$$\psi(C_m \otimes \Gamma) = \Gamma \otimes C_m \quad \text{for} \quad \Gamma \in \mathcal{B}(R) \quad \text{and} \quad m \in (1, 2). \tag{3.19}$$

The elements C_2 is, however, not grouplike

$$\tilde{\Delta}(C_2) \neq C_2 \otimes C_2. \tag{3.20}$$

Taking into account, the braiding properties (3.18), it may be checked that the maps $(\tilde{\Delta}, \tilde{S}, \tilde{\varepsilon})$ preserve the algebra (3.4). In this construction the antipodes of the composite operators are defined by the rule [26-28]

$$\tilde{S}(\Gamma_1 \Gamma_2) = m \circ \psi(\tilde{S}(\Gamma_1) \otimes \tilde{S}(\Gamma_2)) \tag{3.21}$$

where m is the multiplication map. As an example, we demonstrate the property

$$\tilde{S}([\beta, \delta]) = \tilde{S}(2h \gamma \delta - w \alpha \delta). \tag{3.22}$$

Using (3.18) and (3.21), we obtain:

$$\begin{aligned}
\text{l.h.s.} &= m \circ (\gamma^{-1} \psi(\beta \otimes \delta) - \psi(\delta \otimes \beta) \gamma^{-1} + \gamma^{-1} \psi(\delta \otimes \alpha \delta) \gamma^{-1} - \gamma^{-1} \psi(\alpha \delta \otimes \delta) \gamma^{-1}) \\
&= -2h \gamma^{-2} \delta - w \gamma^{-2} \alpha \delta - 2wh \gamma^{-1} + 2wh \\
\text{r.h.s.} &= -2h \gamma^{-2} \delta - w m \circ (\gamma^{-1} \psi(\alpha \otimes \delta) \gamma^{-1}) = \text{l.h.s.}
\end{aligned}$$

We now examine the RE algebra (3.4) in two distinct limits. In the limit $w \rightarrow 0$, the doubly deformed algebra $\mathcal{U}_{h,w}(h(4))$ reduces to the algebra $\mathcal{U}_h(h(4))$ with the standard deformation parameter h . The RE algebra (3.4) now assumes the form of a centrally extended Lie algebra:

$$[\alpha, \beta] = 2h \alpha \gamma, \quad [\alpha, \delta] = 2h \gamma (\gamma - 1), \quad [\beta, \delta] = 2h \gamma \delta, \tag{3.23}$$

where the central element γ equals to a number in an irreducible representation. When $\gamma \neq 0$, the remaining generators (α, β, γ) may be rescaled as follows:

$$J = \frac{\beta}{2h\gamma}, \quad P_+ = \delta, \quad P_- = \alpha. \quad (3.24)$$

The generators (J, P_\pm) then satisfy the algebra:

$$[J, P_\pm] = \pm P_\pm, \quad [P_+, P_-] = \xi, \quad (3.25)$$

where the central charge $\xi = 2h\gamma(1 - \gamma)$ vanishes when $\gamma = 1$. The invariant C_2 of the RE algebra in (3.5) becomes the Casimir element of the algebra (3.25) and reads

$$C_2 = P_- P_+ + \xi J. \quad (3.26)$$

The algebra (3.25) is, in fact, isomorphic to that of the magnetic translation vectors in two dimension, where the central charge term appears because of a magnetic flux perpendicular to the plane. In the $\gamma \neq 1$ case, the RE algebra (3.23) therefore indicates an effective magnetic flux, which may be of relevance in discussing statistical properties. The braided commutations relations between two copies of the algebra (3.23) may be obtained from (3.13) at $w \rightarrow 0$ limit

$$\begin{aligned} \alpha' \alpha &= \alpha \alpha', & \alpha' \beta &= \beta \alpha', & \alpha' \delta &= \delta \alpha', \\ \beta' \alpha &= \alpha \beta' - 2h(\gamma - 1)\alpha', & \beta' \beta &= \beta \beta' - 2h\delta \alpha', & \beta' \delta &= \delta \beta', \\ \delta' \alpha &= \alpha \delta' - 2h(\gamma - 1)(\gamma' - 1), & \delta' \beta &= \beta \delta' - 2h\delta(\gamma' - 1), & \delta' \delta &= \delta \delta'. \end{aligned} \quad (3.27)$$

The central terms γ and γ' commute with all others elements of the extended RE algebra.

In the ‘nonstandard’ $h \rightarrow 0$ limit, only the dimensional deformation parameter w is present. The Hopf algebra $\mathcal{U}_{h,w}(h(4))$ reduces to $\mathcal{U}_w(h(4))$ considered in [13] and the RE algebra (3.4) now reads

$$[\alpha, \beta] = -w \alpha^2, \quad [\alpha, \delta] = -w(\gamma - 1)\alpha, \quad [\beta, \delta] = -w \alpha \delta, \quad (3.28)$$

where the central charge element γ is assumed to have a numerical value. Assuming α to be the invertible, the relations (3.28) may be expressed as a linear algebra as follows. For an invertible α , we choose a set of a new variables

$$X = \alpha, \quad Y = \alpha^{-1}\beta, \quad Z = \delta \quad (3.29)$$

to reexpress (3.28) as a Lie algebra

$$[X, Y] = -wX, \quad [X, Z] = -w(\gamma - 1)X, \quad [Y, Z] = w(\gamma - 1)Y - wZ \quad (3.30)$$

The Casimir element C_2 now reads

$$C_2 = X(Z - (\gamma - 1)Y). \quad (3.31)$$

We distinguish between two cases:

(1). When $\gamma = 1$, we redefine $\hat{J} = \frac{Y}{w}$, $\hat{P}_+ = X$, $\hat{P}_- = Z$. The algebra (3.30) is now isomorphic to the inhomogeneous algebra $iso(1, 1) \simeq t_2 \odot so(1, 1)$:

$$[\hat{J}, \hat{P}_\pm] = \pm \hat{P}_\pm, \quad [\hat{P}_+, \hat{P}_-] = 0 \quad (3.32)$$

with its Casimir invariant $C_2 = \hat{P}_+ \hat{P}_-$.

(2). When $\gamma \neq 1$, we use the variables

$$\hat{J} = \frac{1}{2w}(Y + \frac{Z}{\gamma - 1}), \quad \hat{P}_+ = \frac{X}{w^2(\gamma - 1)}, \quad \hat{P}_- = \frac{1}{w}(Y - \frac{Z}{\gamma - 1}) \quad (3.33)$$

The algebra (3.30) is again isomorphic to $iso(1, 1)$:

$$[\hat{J}, \hat{P}_\pm] = \pm \hat{P}_\pm, \quad [\hat{P}_+, \hat{P}_-] = 0 \quad (3.34)$$

with the corresponding invariant operator

$$C_2 = -w^3(\gamma - 1)^2 \hat{P}_+ \hat{P}_-. \quad (3.35)$$

Thus, in the ‘non-standard’ ($h \rightarrow 0$) limit, the RE algebra (3.28) may be identified with a Lie algebra $iso(1, 1)$ provided α is invertible. As a real form, the algebra $iso(1, 1)$ is isomorphic to (1+1)-dimensional Poincaré algebra. A deformed $iso(1, 1)$ algebra was previously obtained [31] at a contraction limit of the Jordanian deformation of the $sl(2)$ algebra. The unitary representations of the algebra $iso(1, 1)$ are infinite dimensional. The braiding relations for the exchanges between two copies of the RE algebras (3.28) may be read from (3.13) at the ‘nonstandard’ $h \rightarrow 0$ limit:

$$\begin{aligned} \alpha' \alpha &= \alpha \alpha', & \alpha' \beta &= \beta \alpha' - w \alpha \alpha', & \alpha' \delta &= \delta \alpha' - w(\gamma - 1) \alpha', \\ \beta' \alpha &= \alpha \beta' + w \alpha \alpha', & \beta' \beta &= \beta \beta', & \beta' \delta &= \delta \beta' - w \delta \alpha' + w^2(\gamma - 1) \alpha', \\ \delta' \alpha &= \alpha \delta' + w \alpha(\gamma' - 1), & \delta' \beta &= \beta \delta' + w \alpha \delta', \\ \delta' \delta &= \delta \delta' - w \delta(\gamma' - 1) + w(\gamma - 1) \delta'. \end{aligned} \quad (3.36)$$

The RE algebra (3.4) corresponding to the doubly-deformed ($h \neq 0, w \neq 0$) R -matrix (2.12) is essentially nonlinear in character and cannot be reduced to a linear algebra.

4 Conclusion

In summary, we considered the RE algebra for a finite dimensional R -matrix for the doubly-deformed algebra $\mathcal{U}_{h,w}(h(4))$. A representation of the reflection matrix K was constructed

using the matrix generators $L^{(\pm)}$ of the $\mathcal{U}_{h,w}(h(4))$ algebra. The coproduct rules of the $\mathcal{U}_{h,w}(h(4))$ algebra then yields a hierarchy of solutions of the K -matrix. The complementary condition necessary for combining two distinct solutions of the RE algebra yields the braiding relations between the generators of these two algebras. This may be thought as a generalization of Bose-Fermi statistics to braiding statistics, which then may be used to provide a new braided coalgebraic structure to a Hopf algebra generated by the elements of matrix K . The RE algebra and the braiding relations between two copies of the RE algebra depend on both the ‘standard’ and the ‘non-standard’ deformation parameters h and w respectively. For $w \rightarrow 0$ the RE algebra is found to be isomorphic to the algebra of magnetic translation operators on a two-dimensional plane. For the ‘non-standard’ ($h \rightarrow 0$) limit, the RE algebra becomes isomorphic to the inhomogeneous algebra $iso(1, 1)$, which is the Poincaré algebra in $(1 + 1)$ dimension. When both deformation parameters are present ($h \neq 0, w \neq 0$), the RE algebra (3.4) is non-linear in nature.

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